Does Matter Matter?

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Abstract

This paper examines a non-general relativistic cosmological model based on a simple mathematical modification of Minkowski space-time. The model predicts the existence of two mathematically distinct classes of cosmological objects:

- *a class of relatively close objects with apparent magnitudes, etc., similar to those of standard models, but with redshifts limited by 3;*
- *a class of more distant, smaller, brighter objects with unlimited redshifts.*

The model also predicts that the age of the universe is twice the Hubble time, or approximately 25 billion years. Unlike general relativistic cosmologies, this cosmology is independent of the distribution of matter in the universe. Its effects become significant only at cosmological distances.

Key words: age of the universe, angular diameters, apparent magnitudes, cosmological models, matter, number counts, redshifts, relative infinity

1. INTRODUCTION

The standard paradigm for general relativity may be summarized as follows. The invariance of light speed leads to the amalgamation of the seemingly disparate quantities "space" and "time" into a single physical entity "space-time." Its simplest incarnation, Minkowski space-time, explains some phenomena but not others, e.g., the bending of light around massive objects. Minkowski space-time is then modified by assuming that the presence of matter "curves" it; the problem for physics is then to match geometric curvature with an appropriate distribution of matter.

Small-scale physical effects such as the bending of light fit this picture comfortably. Cosmologically, the fit is less certain: although the standard Friedmann–Lemaître–Robertson–Walker models appear to describe the expansion of the universe adequately, there is increasing uneasiness about their fit to the observed matter in the universe (the "missing mass," etc.).

The details of this controversy are too complex and diverse to discuss here. Our purpose is simply to point out the existence of another elementary but relatively unknown mathematical modification of Minkowski space-time which does not invoke the presence of matter, but does produce some surprisingly realistic conclusions about cosmological redshifts. We would like to raise the possibility that this modification, rather than the mass distribution of the universe, may be responsible for some astronomical observations.

The modification? Allow the location of Minkowskian infinity to be relative, i.e., observer-dependent. Now, points at infinity, though somewhat uncommon in physics, abound in classical geometry. To clarify the notion of a relative infinity and its effects on geometric perceptions, and to introduce some of the mathematics used later, we look first at a simple spatial example.

2. RELATIVE INFINITY IN THE EUCLIDEAN PLANE:[1] THE MÖBIUS PLANE

Most readers who have ever encountered a first course in complex variables will have come across the "one-point completion" of the classical Euclidean plane by a point at infinity, i.e., the Möbius plane. Although it is usually discussed in terms of complex numbers, for our purposes, the Möbius plane is more suitably described via the more "old-fashioned" *tetracyclic coordinates*.[2]

Given a rectangular coordinate system $\mathbf{r} = (x, y)$ on the Euclidean plane with the associated dot product $\mathbf{r}\cdot\mathbf{r} = x^2+y^2$, the corresponding tetracyclic coordinates are defined as follows: for any $H \neq 0$,

$$X := Hx, \quad Y := Hy, \quad N := H(\mathbf{r}\cdot\mathbf{r}),$$

i.e.,

$$\mathbf{R} := (X, Y, H, N)^t := H(x, y, 1, \mathbf{r}\cdot\mathbf{r})^t \in \mathbb{R}^4.$$

(Superscripted t's denote transposes.) These coordinates are homogeneous (i.e., proportional quadruples represent the same point) and satisfy the condition

$$[\mathbf{R}, \mathbf{R}] := X^2 + Y^2 - HN = 0,$$

where [,] is the scalar product given by

$$[\mathbf{R}_1, \mathbf{R}_2] := X_1X_2 + Y_1Y_2 - \frac{1}{2}H_1N_2 - \frac{1}{2}H_2N_1$$

for all $\mathbf{R}_i := (X_i, Y_i, H_i, N_i) \in \mathbf{R}^4$, $i = 1,2$. The condition $H \neq 0$ may then be written as $[\mathbf{R}, \mathbf{E}] \neq 0$ for $\mathbf{E} := (0, 0, 0, 1)^t$.

It is easy to show that both lines and circles have tetracyclic equations of the form $[\mathbf{R}, \mathbf{A}] = 0$ for some $\mathbf{A} \in \mathbf{R}^4$ with $[\mathbf{A}, \mathbf{A}] > 0$; more precisely, we have a line whenever $\mathbf{R} = \mathbf{E}$ satisfies this equation. The Euclidean transformations (rotations, reflections, translations, dilatations) have the form $\mathbf{R} \to \mathbf{TR}$, where $\mathbf{T}$ is a nonsingular 4×4 matrix with eigenvector $\mathbf{E}$ that preserves the scalar product $[\ ,\ ]$, i.e., $\mathbf{TE} \propto \mathbf{E}$ and $[\mathbf{TR}, \mathbf{TR}] = [\mathbf{R}, \mathbf{R}]$ for all $\mathbf{R} \in \mathbf{R}^4$.[3]

Now extend the Euclidean plane to the Möbius plane by admitting as points those quadruples $\mathbf{R} = (X, Y, H, N)^t$ with $H = 0$. Since $\mathbf{r} = (X/H, Y/H)^t$ becomes infinite as $H \to 0$, we say that these points are "at infinity." They must still satisfy $[\mathbf{R}, \mathbf{R}] = X^2 + Y^2 - 0N = 0$, i.e., $X = Y = 0$, so there is actually just a single point at infinity: the antipode $\mathbf{E} \propto (0, 0, 0, 1)^t$. Lines may now be thought of as "circles through the point at infinity." The coordinate transformations of our extended plane need no longer have eigenvector $\mathbf{E}$; thus, although they still map lines and circles onto lines or circles, they need not preserve the distinction between the two (since they need not preserve the antipode).

The idea of a relative infinity comes not from the mathematics of the Möbius plane, but rather from a question of interpretation: just what does it mean for a point to be "at infinity"? To make the possible interpretations concrete, we borrow an idea from relativity theory: with each coordinate system, we associate an observer who interprets the geometry of the plane in terms of his or her own coordinate system. We then ask how these observers interpret the antipode.

One interpretation assigns the antipode a special, unique status, preserved by all "allowable" coordinate transformations (i.e., those with eigenvector $\mathbf{E}$). All observers live in the same ordinary Euclidean plane, and see the antipode as an artifact—useful, perhaps, but not really "there." All agree on the distinction between lines and circles—the lines pass through their common antipode. (See Fig. 1. This is the fixed, "unreal" type of infinity found in Penrose diagrams.)

The opposite interpretation denies the antipode any special status whatsoever. All observers live in a complete Möbius plane, with no point in any way distinguishable from the others. Again, the observers agree completely: there are no lines, as distinct from circles. (See Fig. 2. This interpretation of infinity is often found in classical geometry texts.)

The "relativistic" viewpoint lies between these extremes: the antipode retains its special status, but only for individual observers. With respect to his own special antipode, any observer sees an ordinary Euclidean plane, and unambiguously distinguishes lines from circles: the lines are the ones that pass through his antipode. Another observer will generally disagree: although she sees the same objects (Euclidean lines and circles), to *her*, the lines are the ones through *her* antipode (Fig. 3).

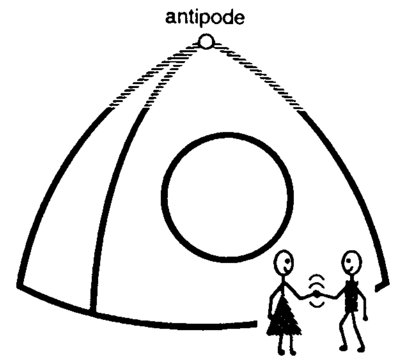

Figure 1. If two observers have the same "special" antipode, then they agree on the distinction between lines and circles.

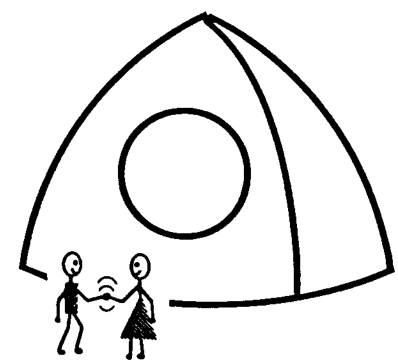

Figure 2. If the observers have *no* distinguished antipode, they all again agree: "lines" do not now exist.

In this case, then, although the two observers share the same universe of points (with two exceptions) and agree that these points constitute a Euclidean plane, they disagree radically about the geometry of that plane. Like simultaneity in special relativity, the location of infinity and the consequent distinction between lines and circles become relative, observer-dependent concepts.

3. RELATIVE INFINITY IN MINKOWSKI SPACE-TIME

The space-time model of this paper bears exactly the same relation to ordinary Minkowski space-time as the Möbius plane (with a relative infinity) does to the Euclidean plane. Cast Minkowski space-time as $\mathbf{R}^4$ with a Lorentzian scalar product (i.e., as a linear geometrical space rather than a differentiable manifold) and define cyclic coordinates by direct analogy to tetracyclic coordinates:[4] given a rectangular coordinate system $\mathbf{r} := (x, y, z, t)^t$ in Minkowski space-time with the corresponding scalar product

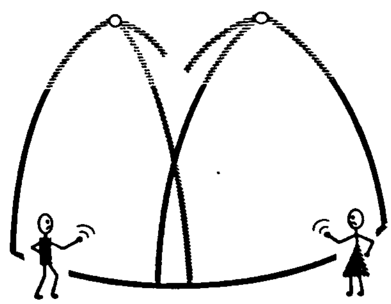

Figure 3. Two observers with *different* antipodes will *disagree* on the distinction between lines and circles.

$$(\mathbf{r},\mathbf{r}):=x^2+y^2+z^2-t^2$$

for any $H \neq 0$, define

$$X:=Hx,\quad Y:=Hy,\quad Z:=Hz,\quad T:=Ht,\quad N:=H(\mathbf{r},\mathbf{r}),$$

i.e.,

$$\mathbf{R}:=H(x,y,z,t,1,(\mathbf{r},\mathbf{r}))^t.$$

These coordinates are homogeneous and satisfy $[\mathbf{R},\mathbf{R}]=0$, where [,] is now the scalar product given by

$$[\mathbf{R}_1,\mathbf{R}_2]:=X_1X_2+Y_1Y_2+Z_1Z_2-T_1T_2-\frac{1}{2}H_1N_2-\frac{1}{2}H_2N_1$$

for all $\mathbf{R}_1 := (X_1, Y_1, Z_1, T_1, H_1, N_1)^t$ and $\mathbf{R}_2 := (X_2, Y_2, Z_2, T_2, H_2, N_2)^t$ in $\mathbf{R}^6$.

Now add the events at infinity: admit as events those sextuples $\mathbf{R} = (X, Y, Z, T, H, N)^t$ with $H = 0$. Then $X^2 + Y^2 + Z^2 - T^2 - 0N = 0$, so in addition to the antipode $\mathbf{E} \propto (0, 0, 0, 0, 0, 1)^t$, we must add other events at infinity: those with $[\mathbf{R}, \mathbf{E}] \equiv -H/2 = 0$. Since for any finite point **A**, the equation $[\mathbf{R}, \mathbf{A}] = 0$ represents the null cone with vertex **A** (easy exercise), the added points are said to constitute the "null cone at infinity." This of course is the standard Minkowskian infinity, except that here, the infinite null cone is not the fixed mathematical artifact of Penrose diagrams, etc., but is equivalent to all other null cones and is interchangeable with them by coordinate transformations (here, multiplications of **R** by nonsingular 6×6 matrices that preserve the scalar product [,]). The location of infinity is now relative.

To examine the physical effects of a relative Minkowskian infinity, we again attach observers to coordinate systems. Each observer is oblivious to his or her own infinity—the observer's space-time consists of the remaining events, and is Minkowskian, at least to the extent that space is Euclidean and light travels in straight lines with speed 1. (The observer does not assume another's proper time to be Minkowskian arclength.) His world-line (his time axis) is a time-like line—as the observer sees it. But just as a relative Euclidean infinity induces some observers to see circles where others see lines, a relative Minkowskian infinity induces some observers to see other observers' world-lines as timelike hyperbolas. (See Fig. 4. These are not the world-lines of observers with constant proper acceleration, since proper time is not arclength. These observers have zero proper acceleration.) We omit the proof; see the next section for a concrete example.

Independent of the location of infinity, then, there is no distinction between time-like lines and time-like hyperbolas: we are forced to treat them identically. The consequences are radical, especially for our usual notion of proper time.

4. RELATIVE INFINITY AND PROPER TIME: SYNCHRONIZING CLOCKS

For an observer using his or her own coordinates, defining proper time is no problem: proper time τ is the time coordinate he assigns to the events on his own world-line (his time axis). The observer can then use proper time to parameterize the world-line: in his own rectangular coordinates, $\mathbf{r}(\tau) = (0, 0, 0, \tau)^t$, or in cyclic coordinates, $\mathbf{R}(\tau) \propto (0, 0, 0, \tau, 1, -\tau^2)^t$. (Note that since $\mathbf{R}(\tau) \propto (0, 0, 0, -\tau^{-1}, -\tau^{-2}, 1)^t$, $\mathbf{R}(\tau) \to \mathbf{E}$ as $\tau \to \pm\infty$, i.e., an observer's world-line contains his or her own antipode as a limiting point.) In his own coordinates, then, an observer's proper time is Minkowskian arclength.

But Minkowskian arclength is not invariant under general cyclic coordinate transformations. In another observer's coordinates, the first observer's proper time is not arclength, but a somewhat more complicated quantity.[5] Only if the two observers are "co-inertial" (i.e., if they share the same antipode) is one observer's proper time the other observer's arclength (up to scale).

The situation becomes more worrisome if we consider two observers with the same world-line. In ordinary (nonextended) Minkowski space-time, the clock readings of two such observers can differ only up to arbitrary choices of clock origin and unit of measurement, i.e., their clocks run uniformly with respect to each other. But if infinity is relative, this no longer holds: the observers may now have different antipodes. In this case, near his own antipode, the first observer's clock reading becomes infinite while the second observer's reading remains finite, and vice versa: their clocks cannot run uniformly with respect to each other. (Their proper times are in fact related by a linear fractional transformation.[5])

This, of course, is heresy. Physically, the unit of time, the second, is defined[1] to be "9,192,631,770 periods of the

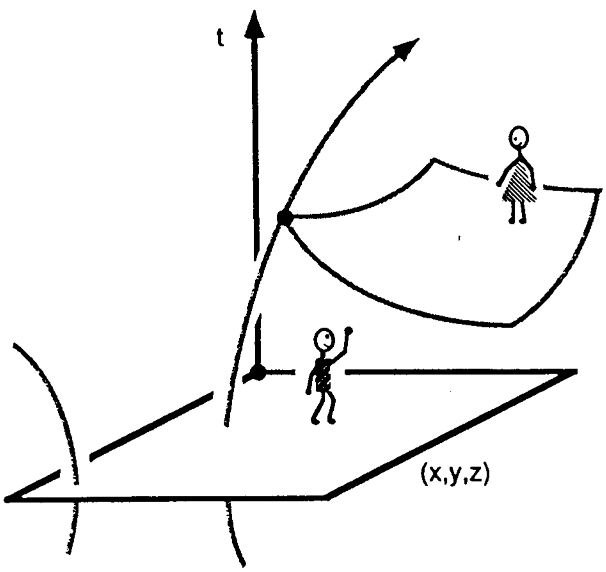

Figure 4. In Minkowski space-time with a relative infinity, one observer's world-line may appear to another observer to be a timelike hyperbola (the asymptotes are null lines).

imperturbed microwave transition between the two hyperfine levels of the ground state of Cs^{133}," and it has been well verified experimentally that other local "atomic clocks" are synchronized to run uniformly with respect to cesium clocks. We appear to have no physical justification for a more general proper time parameter.

But does this agreement among clocks hold universally? It is not clear that a comparison of the rates of physical clocks at cosmological distances is even meaningful. And it is just as easy to believe that atomic clocks here and now agree, not from any intrinsic invariance of proper time, but because, somewhere in the distant past, they were synchronized to agree.

Suppose this is the case; more precisely, suppose that all occupied world-lines contain a common event such that, at this event, the clocks of any two observers are synchronized to

- read 0;
- have rates related by a Minkowskian time dilation (Doppler shift);
- run uniformly with respect to each other.

(Unoccupied world-lines do not present difficulties: a physically defined proper time requires physically present clocks.)

To develop the mathematical implications of these conditions, we first identify two of the observers with ourselves and some distant cosmological object. Our cyclic coordinates **R** and its cyclic coordinates $\overline{\mathbf{R}}$ are related by some coordinate transformation matrix **T**, i.e. $\overline{\mathbf{R}} = \mathbf{TR}$. The common event must have coordinates $\mathbf{B} \propto (0, 0, 0, 0, 1, 0)^t$ in both systems (it is on both world-lines at time 0), so the first synchronization condition implies that **B** is an eigenvector of **T**.

Next, suppose that the object has proper time τ. The second synchronization condition then implies that, in our rectangular coordinates $\mathbf{r} = (\bar{\mathbf{s}}, t) := (x, y, x, t)$, its world-line satisfies

$$\left.\frac{dt}{d\tau}\right|_{t=\tau=0} = \frac{1}{\sqrt{1-\|\bar{v}\|^2}}$$

for

$$\bar{\mathbf{v}} := \left.\frac{d\bar{\mathbf{s}}}{dt}\right|_{t=\tau=0}.$$

The third synchronization condition translates directly into

$$\left.\frac{d^2 t}{d\tau^2}\right|_{t=\tau=0} = 0.$$

From these three mathematical conditions, a straightforward calculation (in the appendix) gives us the object's world-line: up to a Lorentz transformation and a re-orientation of our spatial axes, in our coordinates, the object has world-line

$$x(\tau) = \frac{\delta\tau^2}{1-\delta^2\tau^2}, \quad y(\tau) = z(\tau) = 0, \quad t(\tau) = \frac{\tau}{1-\delta^2\tau^2}$$

for

$$\delta := \frac{1}{2}\left\|\frac{d^2\bar{\mathbf{s}}}{dt^2}\right\|_{t=\tau=0}.$$

Eliminate τ to get the hyperbola

$$t^2 - \left(x + \frac{1}{2}\delta^{-1}\right)^2 = -\left(\frac{1}{2}\delta^{-1}\right)^2$$

illustrated in Fig. 5.

Note first that, since its world-line has two branches, the object appears to be two objects: a "nearby" object

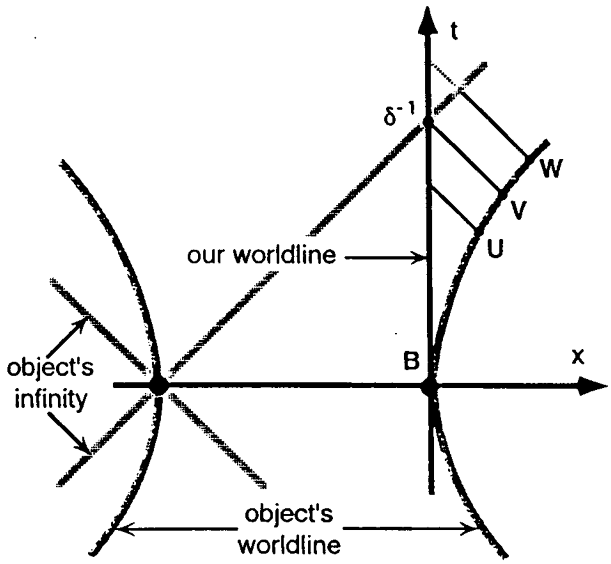

Figure 5. The object's world-line is a hyperbola tangent to our t-axis at event **B**. Its infinity is a null cone with vertex at its antipode $(-\delta^{-1}, 0)$.

(for $|\tau| < \delta^{-1}$) and a "far-away" object (for $|\tau| > \delta^{-1}$). This "splitting" has no physical meaning for the object; it results merely from describing *its* world-line in *our* coordinates. The situation is completely symmetric: to the object, *our* world-line appears split.

Light from the object can thus reach us from two diametrically opposite directions—with one catch: we do not always see it. Light from the near-branch event **U** (Fig. 5) reaches us without complications. But light from event **W** does not: it reaches the object's infinity before it can reach us, and its intensity, which varies by an inverse square law in the object's coordinates, drops off to zero. For the same reason, we cannot observe light from the lower half of the far-branch part of the world-line.

5. REDSHIFTS, MAGNITUDES, ANGULAR DIAMETERS, AND NUMBER COUNTS

Suppose the object emits light at its proper time $\tau = \varepsilon$, which we receive at our proper time $v > 0$ (our "now"). Then

$$v = t(\varepsilon) + |x(\varepsilon)| = \frac{\varepsilon}{1 \mp \delta\varepsilon} \qquad *$$

(the top and bottom signs denote the near and far branches respectively) and the redshift z of the light is given by

$$1 + z = \frac{dv}{d\varepsilon} = \frac{1}{(1 \mp \delta\varepsilon)^2}. \qquad **$$

Light from the "borderline" near-branch event **V** (Fig. 5) arrives at our time $v = \delta^{-1}$, so from *, $\varepsilon = \delta^{-1}/2$, and from **, $z = 3$. All observable near-branch light thus has redshift less than 3. The observable far-branch light can have infinite redshift (as $\varepsilon \to -\delta^{-1}$) or may be blueshifted (if $-\infty < \varepsilon < -2\delta^{-1}$). Note (from Fig. 5) that we cannot observe light from both branches simultaneously.

The apparent magnitude of the incoming light is defined[2] by

$$m := 2.5 \log[(1+z)^2 A(\bar{d})] + \text{constant},$$

where $\bar{d}$ is the distance traveled by the light *in the emitting object's coordinates* and $A(\bar{d})$ is the surface area of a sphere of radius $\bar{d}$. (The constant depends on the type of object.) Since space here is Euclidean, $A(\bar{d}) = 4\pi\bar{d}^2$, so

$$m = 5 \log[(1+z)\bar{d}] + \text{constant}.$$

By symmetry, our world-line in the object's coordinates is

$$\bar{x}(\sigma) = \frac{-\delta\sigma^2}{1 - \delta^2\sigma^2},$$
$$\bar{y}(\sigma) = \bar{z}(\sigma) = 0,$$
$$\bar{t}(\sigma) = \frac{\sigma}{1 - \delta^2\sigma^2}$$

(where σ is our proper time), so $\bar{d} = |\bar{x}(v)|$ and

$$m = 5 \log[(1+z)|\bar{x}(v)|] + \text{constant}.$$

For observable near-branch light, we have $0 < v < \delta^{-1}$, so $\bar{x}(v) < 0$ and

$$-\frac{\bar{x}(v)}{v} = \frac{\delta v}{1 - \delta^2 v^2}.$$

From **,

$$\delta\varepsilon = \frac{\sqrt{1+z} - 1}{\sqrt{1+z}},$$

so from *, $\delta v = \sqrt{1+z}$. Then

$$|\bar{x}(v)| = v \frac{\sqrt{1+z} - 1}{2\sqrt{1+z} - (1+z)},$$

from which

$$m_{\text{near}} = 5\log\left[v\frac{(1+z)+\sqrt{1+z}}{2-\sqrt{1+z}} \right] + \text{constant}.$$

Similarly, for observable far-branch light,

$$m_{\text{far}} = 5\log\left[v\frac{(1+z)+\sqrt{1+z}}{2+\sqrt{1+z}} \right] + \text{constant}.$$

For positive redshifts and comparable objects, these relations are graphed in Fig. 6. The near-branch magnitude curve resembles the corresponding FLRW curves; the far-branch curve shows relatively constant magnitudes.

The *apparent angular diameter* of a cosmological object with diameter A is given[3] by $\theta := A/d$, where d is the distance traveled by the incoming light in *our* coordinates, i.e., $d = |x(\varepsilon)|$.

For observable near-branch light, $x(\varepsilon) > 0$ and

$$\delta\varepsilon = \frac{\sqrt{1+z}-1}{\sqrt{1+z}}$$

as before, so from *,

$$\frac{x(\varepsilon)}{v} = \frac{\delta\varepsilon}{1+\delta\varepsilon} = \frac{\sqrt{1+z}-1}{2\sqrt{1+z}-1},$$

whence

$$\theta_{\text{near}} = Av^{-1}\frac{2\sqrt{1+z}-1}{\sqrt{1+z}-1}.$$

Similarly, for observable far-branch light,

$$\theta_{\text{far}} = Av^{-1}\frac{2\sqrt{1+z}+1}{\sqrt{1+z}+1}.$$

Near-branch apparent angular diameters resemble FLRW ones and decrease with increasing redshift; far-branch objects have relatively constant angular diameters (Fig. 7; the curves are for objects with the same diameter).

For comparable objects (with the same diameter and same redshift $0 \le z < 3$),

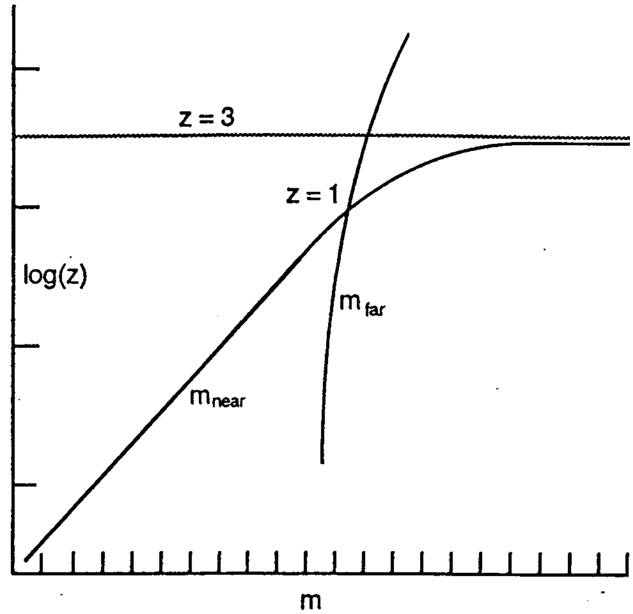

Figure 6. The apparent magnitude relations for near-branch and far-branch objects. The curves cross at $z = 1$, and the near-branch curve is asymptotic to the line $z = 3$.

$$\frac{\theta_{\text{far}}}{\theta_{\text{near}}} = \frac{1+2z-\sqrt{1+z}}{1+2z+\sqrt{1+z}}.$$

This ratio is always less than 1 and decreases as z decreases—it is 0.07 for $z = 0.1$, for example—so for low redshifts, far-branch objects appear significantly smaller than their near-branch counterparts, and more so for lower redshifts.

Expand the near-branch magnitude and angular diameter relations in power series:

$$m_{\text{near}} = 5\log\frac{v}{2} + 5\log z + \frac{15}{4\ln 10}z + (\text{higher order terms in } z) + \text{constant}$$

and

$$\theta_{\text{near}} = \frac{A}{z}\left(\frac{v}{2}\right)^{-1}\left\{1+\frac{5}{4}z + (\text{higher order terms in } z)\right\}.$$

The corresponding relations for FLRW models are[4]

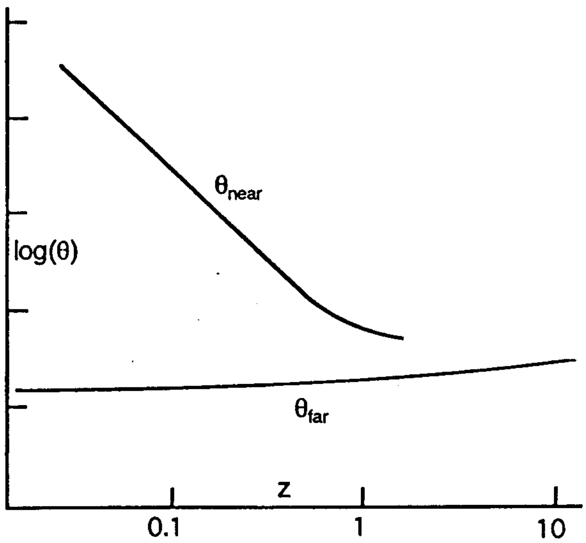

Figure 7. The angular diameter relations for near-branch objects and far-branch objects with positive redshifts.

$$m = 5\log(H_0)^{-1} + 5\log z + \frac{2.5}{\ln 10}(1-q_0)z$$
$$+ (\text{higher order terms in } z) + \text{constant}$$

and

$$\theta = \frac{A}{Z}H_0\left\{1 + \frac{1}{2}(q_0+3)z\right.$$
$$\left. + (\text{higher order terms in } z)\right\},$$

where H_0 and q_0 are the Hubble constant and the deceleration parameter. The formulas agree up to first order for $H_0 = (v/2)^{-1}$ and $q_0 = -1/2$, so if we may assume that the experimental value of H_0 (as calculated to fit the FLRW models) is based on relatively nearby objects, we may conclude that $v = 2H_0^{-1}$, i.e., the age of the universe is twice the Hubble time.

To estimate the number of objects with a given redshift, we look at number counts. Number counts come in several versions; the simplest is $N(<z)$, the number of cosmological objects with redshift less than z. To calculate $N(<z)$, we first need to know how these objects are distributed in space.

Consider the near-branch objects ($\delta < v^{-1}$) located along our positive x-axis. If none has a special place in the universe, then the distribution of objects should look the same to each, so the coordinate transformations relating each object to the next should all be the same. The transformation relating us to the first object (appendix) has matrix

$$C(\delta_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\delta_0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2\delta_0 & 0 & 0 & 0 & 1 & \delta_0^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

for some δ_0, so the transformation relating us to the nth object is $\{C(\delta_0)\}^n = C(n\delta_0)$. It follows that the number n of near-branch objects along our positive x-axis with world-line parameter $\delta = n\delta_0$ or less is proportional to δ. If we look in *all* directions, this number is proportional to δ^3. The relations * and ** give $\delta = v^{-1}\left(\sqrt{1+z}-1\right)$, so

$$N_{\text{near}}(<z) \propto (\sqrt{1+z}-1)^3.$$

From the near-branch apparent magnitude relation,

$$10^{0.6m_{\text{near}}} \propto \left[\frac{\sqrt{1+z}(\sqrt{1+z}-1)}{2\sqrt{1+z}-1}\right]^3,$$

so

$$N_{\text{near}}(<z) \propto 10^{0.6m_{\text{near}}}\left(2 - \frac{1}{\sqrt{1+z}}\right)^3.$$

For small z, this gives $N_{\text{near}}(<z) \propto 10^{0.6m_{\text{near}}}$, as in standard models.

For far-branch objects ($\delta > v^{-1}$), a similar discussion applies (look along our *negative* x-axis). We now have $\delta = v^{-1}(\sqrt{1+z}+1)$, so the number of far-branch objects with redshift less than z is proportional to $\delta^3 - (v^{-1})^3$, i.e.,

$$N_{\text{far}}(<z) \propto (\sqrt{1+z}+1)^3 - 1.$$

This estimate includes all far-branch objects, with redshifts from -1 to $+\infty$. (The near-branch estimate is restricted to $0 \le z < 3$.)

The number $N(=z)$ of objects *at* a given redshift z is the derivative of $N(<z)$. For comparison, the number counts $N(=z)$ for both near- and far-branch objects are graphed in Fig. 8 (for $z > 0$). Note that at any redshift $0 < z < 3$, there are more far-branch objects than near-branch ones.

6. CONCLUSION

How realistic is this model?

1. As noted, the apparent magnitudes, angular diameters, and number counts of near-branch objects resemble those of standard models, which, at least for low redshifts, agree well with astronomical observation.
2. The model predicts that the age of the universe is twice the Hubble time. In light of the recent controversy over the ages of stars vs. the age of the universe,[(5)] this is particularly significant: from the new estimate of $H_0 \approx 80$ km/s Mpc^{-1}, we have a universe with an age of approximately 25 billion years, easily enough to allow for the estimated ages of the stars.
3. Compared to near-branch objects, far-branch objects can have very large redshifts, appear significantly smaller than their near-branch counterparts, and for $z > 1$, appear brighter. These anomalous properties are characteristic of quasars. However, the number counts predict roughly the same number of blueshifted far-branch objects as high redshifted ones ($3 < z < 4$). None has been observed: this apparent dearth of small, bright, blueshifted objects requires explanation.
4. A quasar redshift "drop-off" near $z = 3$ was observed as early as 1972,[6] and is currently a matter of some debate.[(7)] Our model suggests that some quasars with $z \to 3^-$ may be near-branch objects whose infinity we are approaching.
5. The model does not explain the cosmic microwave background radiation. In FLRW models, the back ground radiation is a remnant from the early "unexpanded" universe. Here, our universe is not itself expanding; objects within it just move away from each other. From the geometry of the model (Fig. 5), we receive no radiation from the early universe.

   A *very* tentative candidate for the background radiation is the cumulative result of those photons that reach us after passing their emitter's infinity. (A similar situation occurs in Segal,[(7)] but the model is completely different.)

A final comment: rather than contradicting general relativity, a relative infinity complements it. Its effects become significant only at cosmological distances: for nearby world-lines with nearby antipodes, the effects are negligible, and general relativistic effects due to matter curvature predominate.

So, does matter matter? Certainly, when describing the bending of light, black holes, the geometry of space-time near a star, and the many other small-scale applications of general relativity. But for cosmological redshifts, a relative infinity may perhaps play a role as well.

APPENDIX: DERIVATION OF THE OBJECT'S WORLD-LINE IN OUR COORDINATES

All matrices are 6×6, and are partitioned so that the upper left corner is 3×3.

Since **T** is the matrix of a cyclic coordinate transformation (which preserves the scalar product [,]), it must satisfy $\mathbf{T}^t\mathbf{G}\mathbf{T} = \mathbf{G}$ for

$$\mathbf{G} := \begin{bmatrix} \mathbf{I}_3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1/2 \\ 0 & 0 & -1/2 & 0 \end{bmatrix}.$$

Set

$$\mathbf{T} := \begin{bmatrix} A & B & C & D \\ E^t & F & G & I \\ J^t & K & L & M \\ O^t & P & Q & R \end{bmatrix}.$$

Then the relation $\mathbf{T}^t\mathbf{G}\mathbf{T} = \mathbf{G}$ gives ten identities:

1. $2A^tA - 2E^tE - OJ^t - JO^t = 2I_3$,
2. $2A^tB - 2FE - OK - JP = 0$,
3. $2A^tC - 2GE - OL - JQ = 0$,
4. $2A^tD - 2IE - OM - JR = 0$,
5. $B^tB - F^2 - KP = -1$,
6. $2B^tC - 2FG - PL - KQ = 0$,
7. $2B^tD - 2FI - PM - KR = 0$,
8. $C^tC - G^2 - QL = 0$,
9. $2C^tD - 2GI - QM - LR = -1$,
10. $D^tD - I^2 - MR = 0$.

Since $\mathbf{TB} \propto \mathbf{B}$ for $\mathbf{B} = (0, 0, 0, 0, 1, 0)^t$, C, G, and Q vanish. Identity 9 gives $L = R^{-1}$. Then 3 and 6 show that O and P vanish. Identities 4, 7, and 10 give $\mathbf{T} = \mathbf{CI}$ for

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_3 & 0 & 0 & D \\ 0 & 1 & 0 & I \\ 2R^{-1}D^t & -2R^{-1}I & R^{-1} & R^{-1}[D^tD - I^2] \\ 0 & 0 & 0 & R \end{bmatrix}$$

and

$$\mathbf{I} := \begin{bmatrix} A & B & 0 & 0 \\ E^t & F & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Identities 1, 2, and 5 now show that $\mathbf{I}^t\mathbf{G}\mathbf{I} = \mathbf{G}$, i.e., **I** is the matrix of a cyclic coordinate transformation, and thus so is $\mathbf{C} = \mathbf{T}\mathbf{I}^{-1}$.

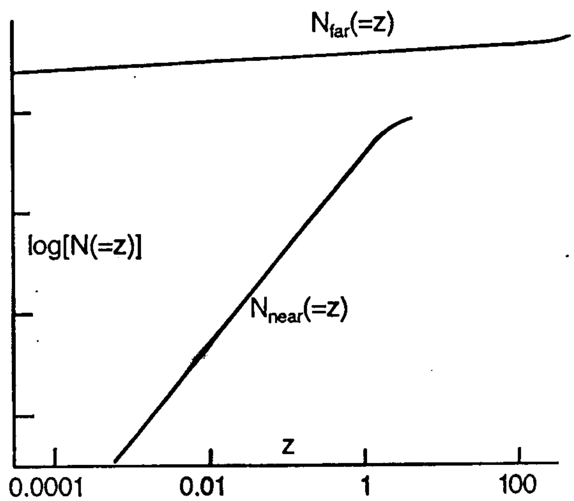

Figure 8. The number counts for near-branch objects and far-branch objects with positive redshifts.

Matrix **I** gives a Lorentz transformation (in rectangular coordinates, its matrix is the upper left 4 × 4 corner of **I**). Up to this Lorentz transformation, our cyclic coordinate transformation is then given by **C**, so in our cyclic coordinates, the object's world-line is effectively

$$\mathbf{R}(\tau) \propto \mathbf{C}(0,0,0,\tau,1,-\tau^2)^t,$$

or, in rectangular form,

$$\bar{\mathbf{s}}(\tau) = R\frac{-D\tau^2}{-2I\tau+1-[D^tD-I^2]\tau^2},$$
$$t(\tau) = R\frac{\tau - I\tau^2}{-2I\tau+1-[D^tD-I^2]\tau^2}.$$

The second synchronization relation

$$\frac{dt}{d\tau}\bigg|_{t=\tau=0} = \frac{1}{\sqrt{1-\bar{\mathbf{v}}^2}} \text{ for } \bar{\mathbf{v}} := \frac{d\bar{\mathbf{s}}}{dt}\bigg|_{t=\tau=0}$$

may be rewritten as

$$\frac{d\bar{\mathbf{s}}}{d\tau}\cdot\frac{d\bar{\mathbf{s}}}{d\tau} - \left(\frac{dt}{d\tau}\right)^2 = -1 \text{ at } t = \tau = 0,$$

which, upon direct calculation, simplifies to $R^2 = 1$. Since **C** and −**C** represent the same cyclic coordinate transformation, we may take $R = 1$. The third synchronization relation

$$\frac{d^2t}{d\tau^2}\bigg|_{t=\tau=0} = 0$$

then gives $I = 0$.

Finally, if we reorient our spatial axes so that our x-axis is parallel to

$$\frac{d^2\bar{\mathbf{s}}}{dt^2}(0) = -2D,$$

then

$$D = (-\delta,0,0)^t \text{ for } \delta := \frac{1}{2}\left\|\frac{d^2\bar{\mathbf{s}}}{dt^2}\right\|_{t=\tau=0}$$

and we have the world-line

$$x(\tau) = \frac{\delta\tau^2}{1-\delta^2\tau^2},$$
$$y(\tau) = z(\tau) = 0,$$
$$t(\tau) = \frac{\tau}{1-\delta^2\tau^2},$$

as required.

Note that the final form of the matrix **C** is

$$\mathbf{C}(\delta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\delta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -2\delta & 0 & 0 & 0 & 1 & \delta^2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

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Résumé

Cet article analyse un modèle cosmologique basé sur une transformation mathématique simple de l'espace de Minkowski plutôt que sur les concepts de la relativité générale. Ce modèle prédit l'existence de deux classes distinctes d'objets cosmologiques :

- *Une classe d'objets relativement proches, possédant des magnitudes apparentes, etc. Ces objets sont semblables à ceux des modèles courants, mais leur décalage vers le rouge est limité à 3;*
- *Une seconde classe d'objets, plus distants, plus petits et plus brillants. Ces derniers possèdent un décalage illimité vers le rouge.*

Le modèle suggère également que l'âge de l'univers est d'environ 25 milliards d'années soit, deux fois le temps de Hubble. Contrairement aux théories cosmologiques basées sur la relativité générale, le présent modèle est indépendant de la distribution de la matière dans l'univers et ses effets ne deviennent importants qu'à l'échelle cosmologique.

Endnotes

1 For more on this topic, see J.A. Lester, J. Geom. **46** (1993), pp. 92–118

2 See, for example, J.L. Coolidge, *A Treatise on the Circle and the Sphere* (Chelsea, New York, 1971; originally Oxford, 1916).

3 Some examples:
For a 2 × 2 orthogonal matrix **A**, the isometry $\mathbf{r} \to A\mathbf{r}$ has matrix

$$\mathbf{T} = \begin{bmatrix} A & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For a vector $\mathbf{b} \in \mathbf{R}^2$, the translation $\mathbf{r} \to \mathbf{r} + \mathbf{b}$ has matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_2 & \mathbf{b} & 0 \\ 0 & 1 & 0 \\ 2\mathbf{b}^t & (\mathbf{b},\mathbf{b}) & 1 \end{bmatrix}.$$

For a scalar $\lambda \neq 0$, the dilatation (scale change) $\mathbf{r} \to \lambda\mathbf{r}$ has matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

These matrices all have eigenvector **E**. The inversion in the unit circle $(\mathbf{r} \to (\mathbf{r}, \mathbf{r})^{-1}\mathbf{r})$ has matrix

$$\mathbf{T} = \begin{bmatrix} \mathbf{I}_2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

which does not have eigenvector **E**; it interchanges **E** and the origin (0, 0, 1, 0).

4 Cyclic coordinates in Minkowski space-time may also be developed axiomatically. Assume that each observer's finite universe is a copy of Minkowski space-time, but that the overlap between any two such universes need only be a (mathematically) open subset. If two observers always agree on whether or not a light signal can pass between each pair of common events, then without further assumptions, the transformation relating them must be cyclic. For details, see A.D. Alexandrov, Vestnik Leningrad Univ. Math. **11** (1976), pp. 95–100, or J.A. Lester, Ann. Discrete Math. **18** (1983), pp. 567–574. Another space-time model with a relative infinity can be similarly developed using copies of de Sitter space-time: see J.A. Lester, Astronomy and Astrophysics **207** (1993), pp. 231–248.

5 Further details of this and other conclusions may be found in J.A. Lester, Il Nuovo Cimento **72B** (1982), pp. 261–272, and Il Nuovo Cimento **73B** (1983), pp. 139–149.

6 See for example the following papers in *The Space Distribution of Quasars*, D. Crampton, ed., ASP Conference Series **21** (1991): M. Schmidt, D. Schneider, J. Gunn, pp. 109–114; M. Irwin, R. McMahon, C. Hazard, pp.117–126; S.J. Warren, P.C. Hewett, P.S. Osmer, pp. 139–148.

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4. S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).
5. W.L. Freedman et al., Nature **371** (1994), p. 757.
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